Exercise 1

a1)

$$\frac{dx}{dt} = 0$$

$$(1-x) - xy = 0$$

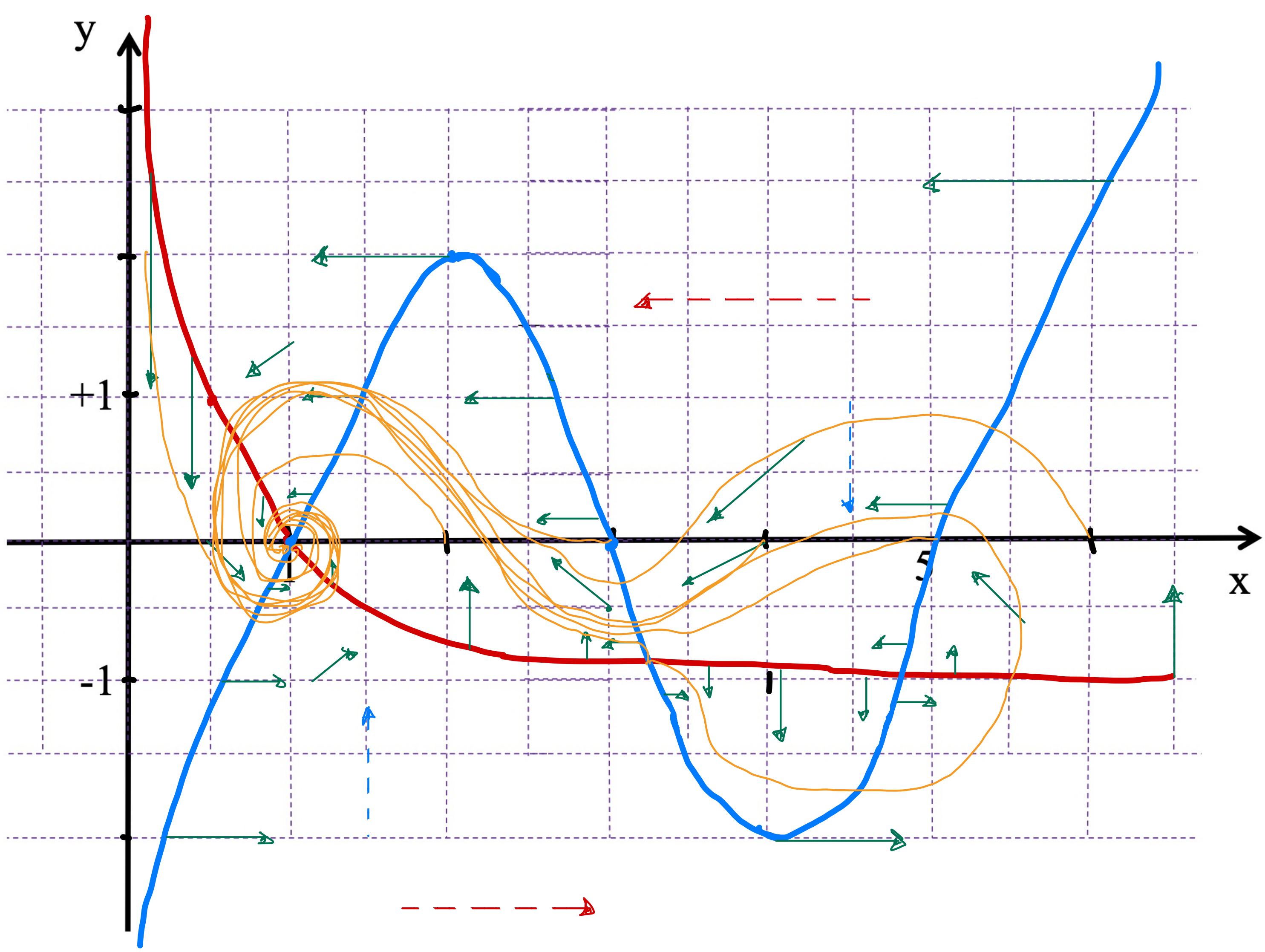
$$xy = 1 - x \qquad y = \frac{1-x}{x} = \frac{1}{x} - 1$$

a2)

$$\frac{dy}{dt} = 0$$

$$y = F(x) = \begin{cases} -2(x-1)(x-3) & \text{iff } x \leq 3 \\ 2(x-3)(x-5) & \text{iff } x > 3 \end{cases}$$

b) $\frac{dx}{dt} = 0 \quad \frac{dy}{dt} = 0$



First bronch

$F(x \leq 3) =$

$-2(x-1)(x-3) =$

$-2(x^2 - 3x - x + 3) =$

$-2(x^2 - 4x + 3) =$

$-2x^2 + 8x - 6$

$\frac{dF}{dx} = -4x + 8 = 0$

$x = 2$

$y = 2$

Second bronch

$F(x>3) =$

$2(x-3)(x-5) =$

$2(x^2-5x-3x+15) =$

$2x^2-16x+30$

$\frac{dF}{dx} = 4x - 16 = 0$

$x = 4$

$y = -2$

c) In $(3, -0.5)$

$\frac{dx}{dt} = -2 + 1.5 = -0.5$

$\frac{dy}{dt} = 0.5 + 0$

(See plot!)

In $(4, 0)$

$\frac{dx}{dt} = -3$

$\frac{dy}{dt} = -2$

e) Left-most stable point in $(1, 0)$

$$\rightarrow \quad \frac{(1-x) - xy}{\tau_x \, dx} = \frac{-1-y}{\tau_x}$$

$$\rightarrow \quad \frac{(1-x) - xy}{\tau_x \, dy} = \frac{-x}{\tau_x}$$

$$\rightarrow \quad \frac{-y + F(x)}{\tau_y \, dx} = \frac{-2x^2 + 8x - 6}{\tau_y \, dx} =$$

$$= \frac{-4x + 8}{\tau_y}$$

$$\rightarrow \quad \frac{-y + F(x)}{\tau_y \, dy} = -\frac{1}{\tau_y}$$

In summary

$$J = \begin{pmatrix} \dfrac{-1-y}{t_x} & -\dfrac{x}{t_x} \\[2em] \dfrac{-4x+\rho}{t_y} & -\dfrac{1}{t_y} \end{pmatrix}$$

Plug the values!

$$J = \begin{pmatrix} -\dfrac{1}{t} & -\dfrac{1}{t} \\[1.5em] \dfrac{4}{t} & -\dfrac{1}{t} \end{pmatrix} =$$

$$\frac{1}{t} \begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix}$$

Eigenvalues time!

$$\det \begin{pmatrix} -1-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix} = (-1-\lambda)^2 + 4$$

$$(-1-\lambda)^2 + 4 = 0$$

$$1 + \lambda^2 + 2\lambda + 4 = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} =$$

$$-1 \pm 2i$$

Eigenvalues are $\underline{-1 \pm 2i}$

t

Always negative real part!

f) UNCLEAR!

1) How can we tell if it's a saddle point without computing eigenvalues? (I.e., is the third point a saddle?)

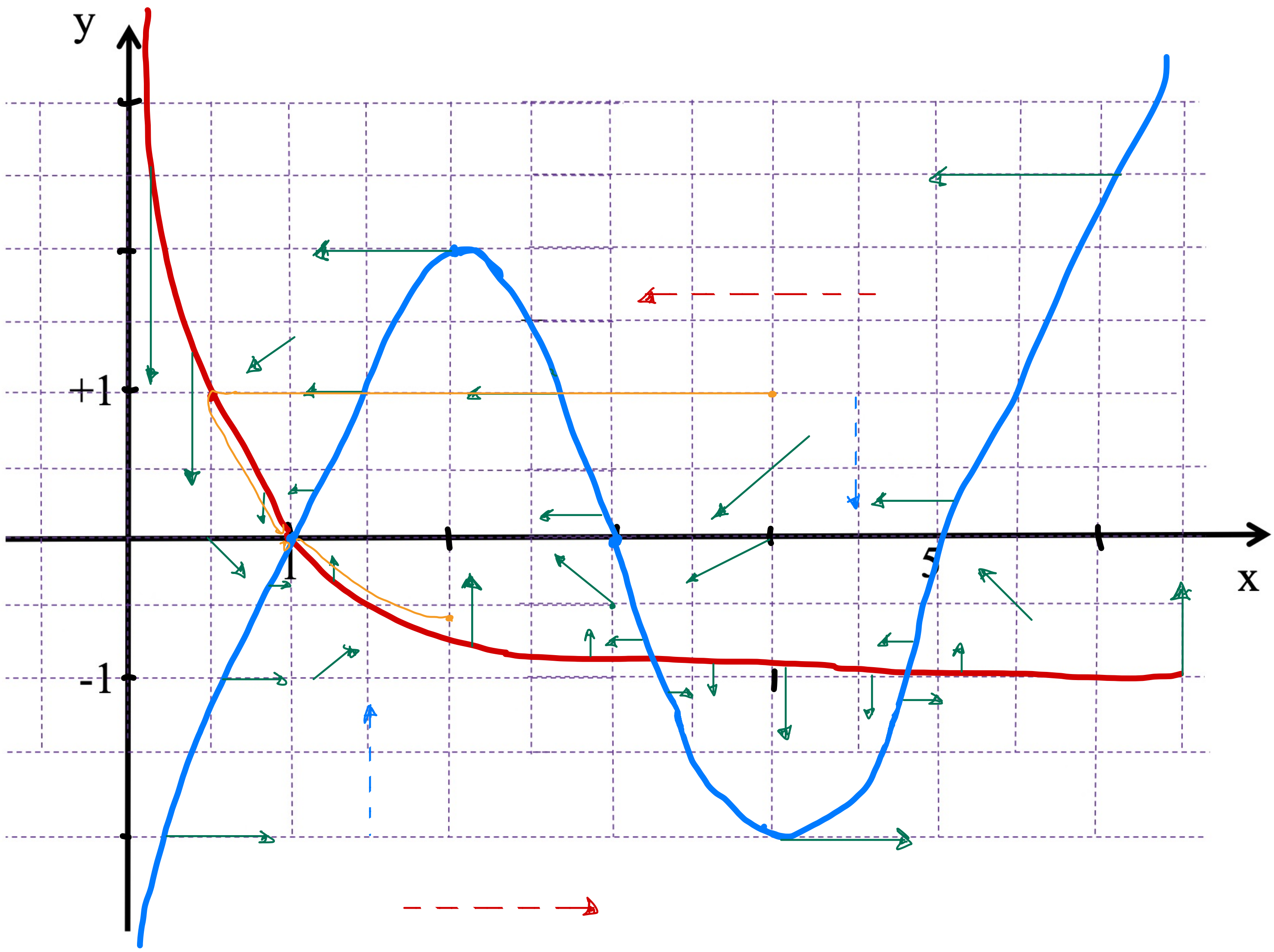
2) How can you END in a saddle point??

g) (see plot)

h) (see plot)

i) (see plot)

j)



Exercise 2

a)

$$t_I \frac{dh_i}{dt} = -h_i + RI(t) + t_I w_{II} A(t)$$

b)

$$h_i(t) = \int_0^t \left(RI(t') + t_I w_{II} A(t')\right) e^{-(t-t')/t_I} dt'$$

Like in the first exercise session!
Think of it as many small charges deposited a step at a time!

c) The same. they both start from the same value and receive the same exact input.
(Yes they fire randomly but this has a global, not an individual effect!)

d) All $h_k = h$ so we only study one neuron (see step c)

We have

$$A(t) = g(\underset{\substack{\nearrow \\ h>0 \text{ always}}}{h(t)}) = \frac{h}{\tau_I h_0}$$

By what we wrote in d) we have

$$\tau_I \frac{dh}{dt} = -h + RI(t) + \tau_I w_{II} A(t) =$$

$$= -h + RI(t) + \cancel{\tau_I} w_{II} \frac{h}{\cancel{\tau_I} h_0}$$

$$= -\left(1 - \frac{w_{II}}{h_0}\right) h + RI(t)$$

$$\frac{\tau_I}{\left(1 - \frac{w_{II}}{h_0}\right)} \frac{dh}{dt} = -h + \frac{RI(t)}{\left(1 - \frac{w_{II}}{h_0}\right)}$$

Integrating (see discussion in b) we get

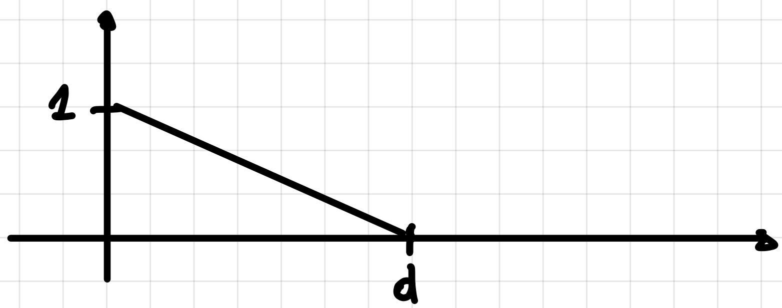
$$h(t) = \int_0^t \frac{R\,I(t')}{\left(1 - \frac{w_{II}}{h_0}\right)} \exp\left(- \frac{(t-t')\left(1 - \frac{w_{II}}{h_0}\right)}{t_I} \right) dt'$$

And $A(t)$ immediately follows from

$$A(t) = \frac{h(t)}{T_I\, h_0} \qquad (h(t) > 0 \ \forall t)$$

Exercise 3

$$\alpha(\underbrace{s}_{\text{time}}|\underbrace{d}_{\text{duration}}) = \overbrace{H(s)\,H(d-s)}^{\text{Box }[0,d]}\,\overbrace{\left[1-(s/d)\right]}^{\text{Ramp down}}$$



a) Each spike deposits $d_i/2$ charge (it's a triangle!) at a rate ν_i.

$$\langle I \rangle = \frac{k\left(\nu_1 w_1 d_1 + \nu_2 w_2 d_2\right)}{2}$$

b)

$$\langle I \rangle = \cancel{1000}\left(5\cdot\cancel{10^{-3}}\,\cancel{ms^{-1}}\cdot 1\mu A\cdot\cancel{1ms} + 2.5\cdot\cancel{10^{-3}}\,\cancel{ms^{-1}}\cdot(-0.2\mu A)\cdot 10\,\cancel{ms}\right)/2 =$$

$$(5\mu A - 5\mu A)/2 = 0\mu A$$

c)

$$\langle I \rangle = (5\mu A - 20\mu A)/2 = -7.5\mu A$$

d) $$Var(I) = E(I^2) - \underbrace{E(I)^2}_{=0 \text{ by hypothesis}}$$

What is the distribution for I?

→ At any time, the number S of ongoing spikes is distributed with

$$S \sim \mathrm{Pois}[\nu_i d_i]$$

→ The current variance within a single spike for a single neuron is given by

$$Var(I|S=1) = E(I^2) - E(I)^2 =$$

$$\frac{1}{w_i}\int_0^{w_i} x^2\,dx - \left(\frac{w_i}{2}\right)^2 =$$

$$= \frac{1}{w_i}\left[\frac{1}{3}x^3\right]_0^{w_i} - \left(\frac{w_i}{2}\right)^2 = \frac{1}{3}w_i^2 - \frac{1}{4}w_i^2 =$$

$$\frac{1}{12}w_i^2$$

For $\bar{S}$ spikes, we just sum the variance:

$$\mathrm{Var}(I \mid \bar{S}) = \bar{S} \, \frac{1}{12} w_i^2$$

Now we need to use the law of total variance:

$$\mathrm{Var}(I) = E[\mathrm{Var}(I \mid S)] + \mathrm{Var}(E[I \mid S])$$

$$= E\left(S \frac{1}{12} w_i^2\right)$$

$$= \frac{1}{12} w_i^2 E(S) =$$

$$= \frac{1}{12} w_i^2 \nu_i d_i$$

$$E[I \mid \bar{S}] = \frac{1}{2} w_i \bar{S}$$

$$= \mathrm{var}\left(\frac{1}{2} w_i S\right) = \frac{1}{4} w_i^2 \, \mathrm{var}(S)$$

$$= \frac{1}{4} w_i^2 \nu_i d_i$$

Is this even True??

$$\mathrm{var}(I) = \frac{1}{3} \nu_i d_i w_i^2$$

YES!!

Numerically verified!!

All of the above for a single neuron.
Variance sums again for all neurons:

$$\mathrm{Var}(I) = K/3 \left(\nu_1 d_1 w_1^2 + \nu_2 d_2 w_2^2 \right)$$

e) i) Standard deviation doubles

ii) Standard deviation grows by $\sqrt{2}$

iii) Standard deviation grows by $\sqrt{2}$

Exercise 4

d)

$$\tau_x \frac{dx}{dt} = \frac{3}{7}(1-x) - y$$

$$\frac{7}{3} \tau_x \frac{dx}{dt} = 1 - x - \frac{7}{3} y$$

$$= \left(1 - \frac{7}{3} y\right) - x$$

$\ll \tau_y$

We can set

$$x(t) = 1 - \frac{7}{3} y(t)$$

And replace

$$\tau_y \frac{dy}{dt} = -y + F\left(1 - \frac{7}{3} y\right)$$

$$t_y \frac{dy}{dt}(\varepsilon \ll 1) = -\varepsilon + F(1 - 7/3\,\varepsilon) =$$

$$= -\varepsilon - 2(\cancel{1} - 7/3\,\varepsilon - \cancel{1})(1 - 7/3\,\varepsilon - 3) =$$

$$= -\varepsilon + \frac{14}{3}\varepsilon\left(-7/3\,\varepsilon - 2\right) \cong -\varepsilon - \frac{28}{3}\varepsilon \cong -\varepsilon$$

$$t_y \frac{dy}{dt}(0) = F(1) = 0$$

$$t_y \frac{dy}{dt}\left(-\frac{3}{7}\right) = \frac{3}{7} + F\left(1 + 7/3\,\frac{3}{7}\right) = \frac{3}{7} + F(2) =$$

$$\frac{3}{7} - 2 \cdot 1 \cdot (-1) = \frac{3}{7} + \frac{14}{7} = +\frac{17}{7}$$

$$t_y \frac{dy}{dt}\left(-9/7\right) = \frac{9}{7} + F\left(1 + \frac{7}{3}\,\frac{9}{7}\right) = \frac{9}{7} + F(4)$$

$$= 9/7 + 2 \cdot 1 \cdot (-1) = \frac{9}{7} - \frac{14}{7} = -5/7$$

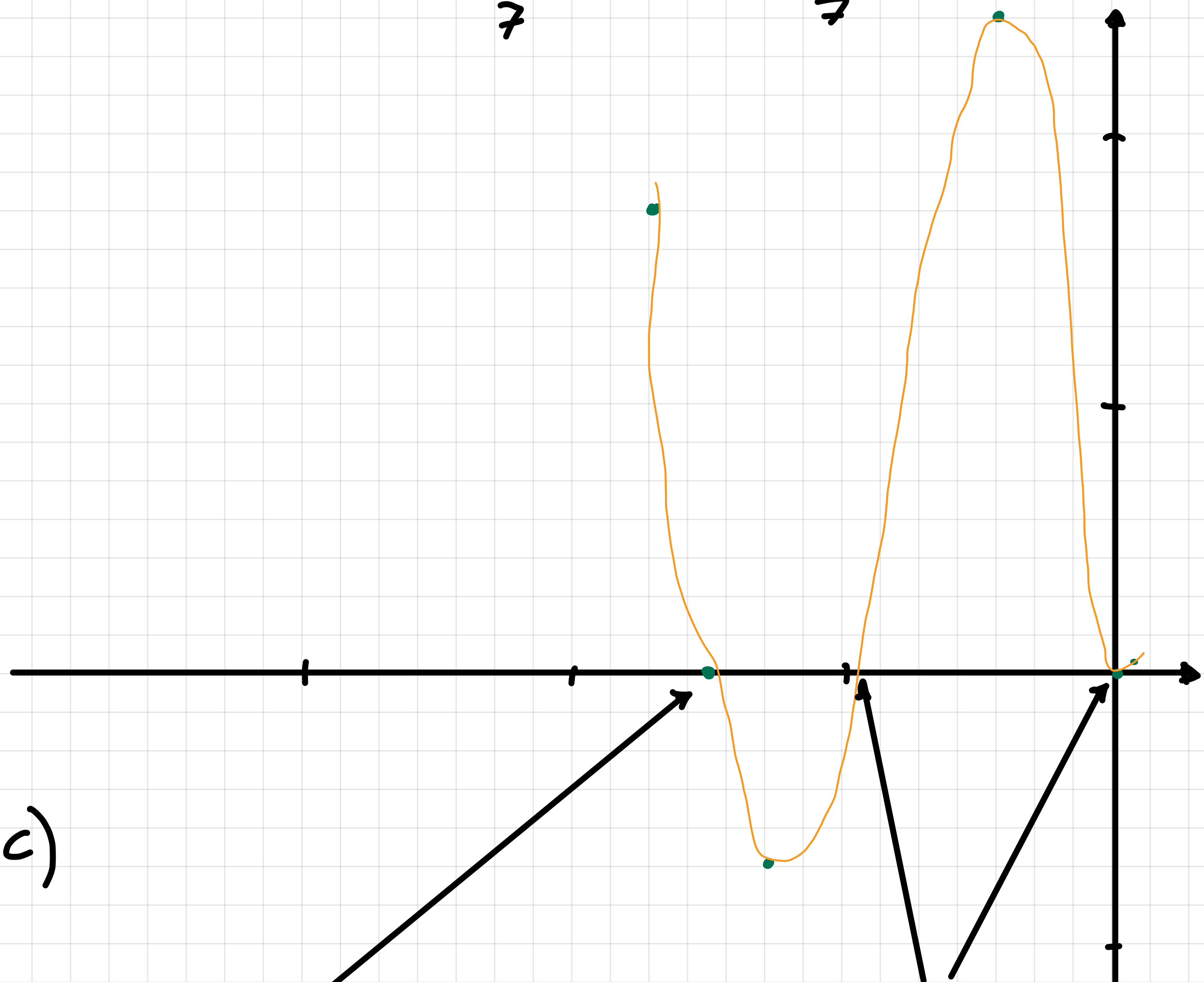
$$t_y \frac{dy}{dt}\left(-\frac{3}{2}\right) = \frac{3}{2} + F\left(1 + \frac{7}{3}\,\frac{3}{2}\right) = \frac{3}{2} + F\left(\frac{9}{2}\right)$$

$$\frac{3}{2} + 2\left(\frac{9}{2} - \frac{6}{2}\right)\left(\frac{9}{2} - \frac{10}{2}\right) = \frac{3}{2} + \cancel{2}\,\frac{3}{\cancel{2}}\left(-\frac{1}{2}\right) = 0$$

$$t_y \frac{dy}{dt}\left(-\frac{12}{7}\right) = \frac{12}{7} + F\left(1 + \frac{\cancel{7}}{3}\frac{12}{\cancel{7}}\right) = \frac{12}{7} + F(5)$$

$$= \frac{12}{7} + 0 = \frac{12}{7}$$

c)



stable

(Perturbations are corrected)

unstable

(Perturbations are amplified: in both directions (1st) or on the right (2nd))